➢Koraki pri reševanju z MKE:

5) določitev začetnih, robnih in obremenitvenih pogojev

• toplotni problem - <u>časovno ustaljen prevod toplote v volumnu:</u>

• diferencialna enačba problema

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+Q=0$$

• osnovna integralska enačba problema

$$\int_{\Omega}\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+Q\right] f(x,y,z)\ \mathrm{d}\Omega=0$$

- preureditev osnovne integralske enačbe problema

$$\int_{\Omega}\left[\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right)f+\frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right)f+\frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right)f\,\right]\mathrm{d}\Omega+\int_{\Omega}Q\,f\,\mathrm{d}\Omega=0$$

- upoštevajoč matematično zvezo

$$\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right)f=\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\,f\right)-k\,\frac{\partial T}{\partial x}\,\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right)f=\frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\,f\right)-k\,\frac{\partial T}{\partial y}\,\frac{\partial f}{\partial y}$$

$$\frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right)f=\frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\,f\right)-k\,\frac{\partial T}{\partial z}\,\frac{\partial f}{\partial z}$$

lahko osnovno integralsko enačbo problema zapišemo v obliki

$$k\int_{\Omega}\left[\frac{\partial T}{\partial x}\,\frac{\partial f}{\partial x}+\frac{\partial T}{\partial y}\,\frac{\partial f}{\partial y}+\frac{\partial T}{\partial z}\,\frac{\partial f}{\partial z}\right]\mathrm{d}\Omega=$$

$$=\int_{\Omega}\left[\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\,f\right)+\frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\,f\right)+\frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\,f\right)\right]\mathrm{d}\Omega+\int_{\Omega}Q\,f\,\mathrm{d}\Omega$$

• z uporabo Green-ovega teorema lahko integral po volumnu

$$I_\Omega = \int_\Omega \left[\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\, f \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\, f \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\, f \right) \right] \mathrm{d}\Omega$$

prevedemo v integral po površini, ki ta volumen omejuje:

$$I_\Omega = I_\Gamma = \int_\Gamma \left[\left(k \frac{\partial T}{\partial x}\, f \right) n_\mathrm{x} + \left(k \frac{\partial T}{\partial y}\, f \right) n_\mathrm{y} + \left(k \frac{\partial T}{\partial z}\, f \right) n_\mathrm{z} \right] \mathrm{d}\Gamma$$

• upoštevajoč Fourierjev zakon o prevajanju toplote

$$q_\mathrm{n} = -k \frac{\partial T}{\partial n}$$

lahko integral po površini zapišemo

$$I_\Gamma = \int_\Gamma \left[\left(k \frac{\partial T}{\partial x}\, f \right) n_\mathrm{x} + \left(k \frac{\partial T}{\partial y}\, f \right) n_\mathrm{y} + \left(k \frac{\partial T}{\partial z}\, f \right) n_\mathrm{z} \right] \mathrm{d}\Gamma =$$

$$= -\int_\Gamma [\, q_\mathrm{x}\, n_\mathrm{x} + q_\mathrm{y}\, n_\mathrm{y} + q_\mathrm{z}\, n_\mathrm{z} \,]\; f \;\mathrm{d}\Gamma$$

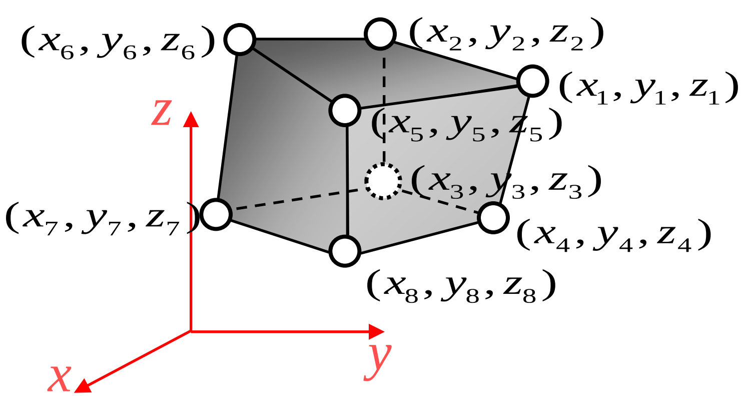
• šibka oblika integralske enačbe problema, ki predstavlja izhodišče MKE

$$k\int_{\Omega}\left[\frac{\partial T}{\partial x}\frac{\partial f}{\partial x}+\frac{\partial T}{\partial y}\frac{\partial f}{\partial y}+\frac{\partial T}{\partial z}\frac{\partial f}{\partial z}\right]\mathrm{d}\Omega=$$

$$=-\int_{\Gamma}[q_{\mathrm{x}}\,n_{\mathrm{x}}+q_{\mathrm{y}}\,n_{\mathrm{y}}+q_{\mathrm{z}}\,n_{\mathrm{z}}]\;f\,\mathrm{d}\Gamma+\int_{\Omega}Q\,f\,\mathrm{d}\Omega$$

• interpolacija temperaturnega polja po območju KE

$$T(x,y,z)\approx\hat{T}(x,y,z)=\sum_{\mathrm{j=1}}^{\mathrm{N_v}}T_{\mathrm{j}}\;\psi_{\mathrm{j}}(x,y,z)$$



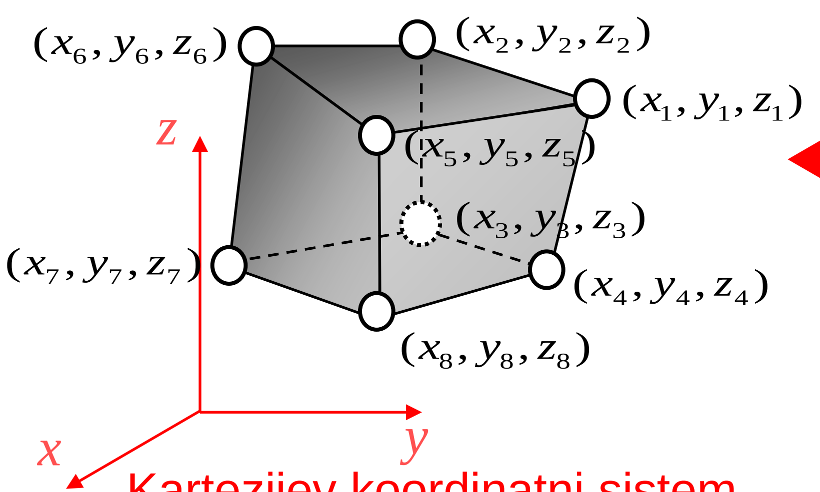
• izoparametrični KE

• interpolacija geometrije KE

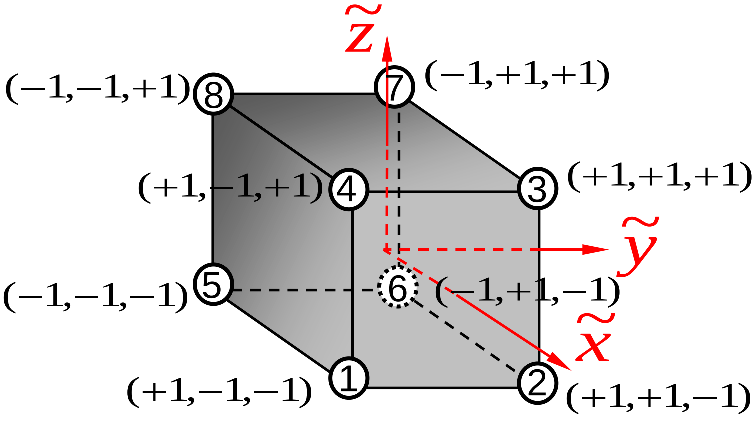
$$x = x(\tilde{x},\tilde{y},\tilde{z}) = \sum_{j=1}^{N_v} x_j \ \tilde{\psi}_j(\tilde{x},\tilde{y},\tilde{z})$$

$$y = y(\tilde{x},\tilde{y},\tilde{z}) = \sum_{j=1}^{N_v} y_j \ \tilde{\psi}_j(\tilde{x},\tilde{y},\tilde{z})$$

$$z = z(\tilde{x},\tilde{y},\tilde{z}) = \sum_{j=1}^{N_v} z_j \ \tilde{\psi}_j(\tilde{x},\tilde{y},\tilde{z})$$



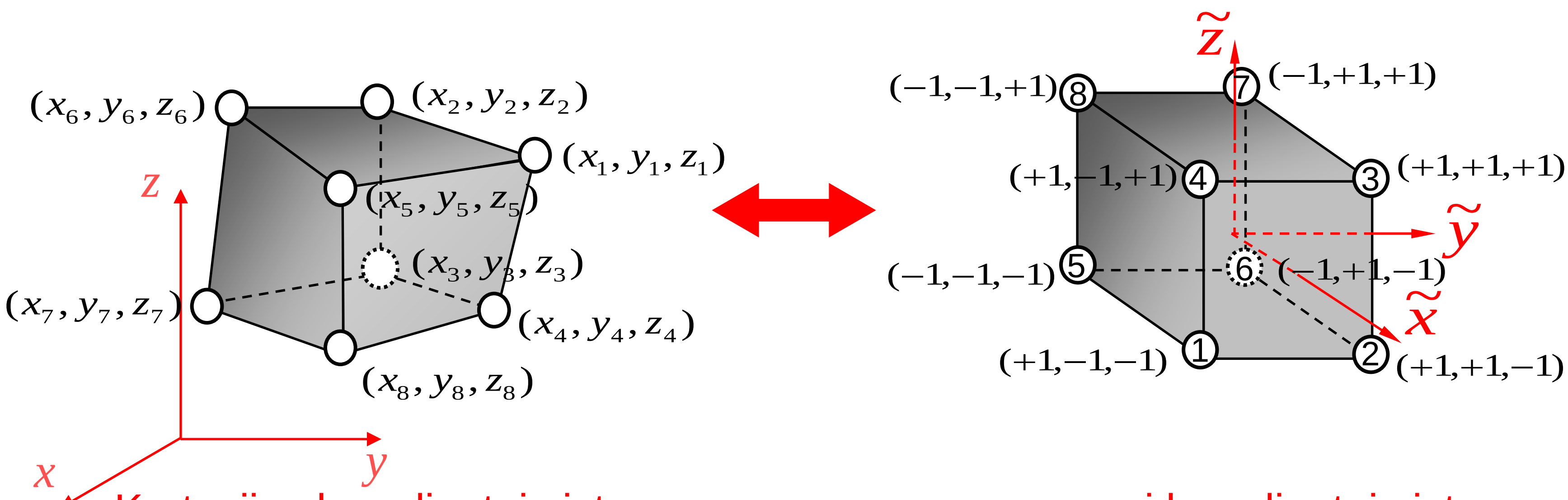
Kartezijev koordinatni sistem



naravni koordinatni sistem

- interpolacija temperaturnega polja po območju KE

$$\hat{T}(x,y,z)=\tilde{T}(\tilde{x},\tilde{y},\tilde{z})=\sum_{j=1}^{N_v} T_j\ \tilde{\psi}_j(\tilde{x},\tilde{y},\tilde{z})$$



Kartezijev koordinatni sistem

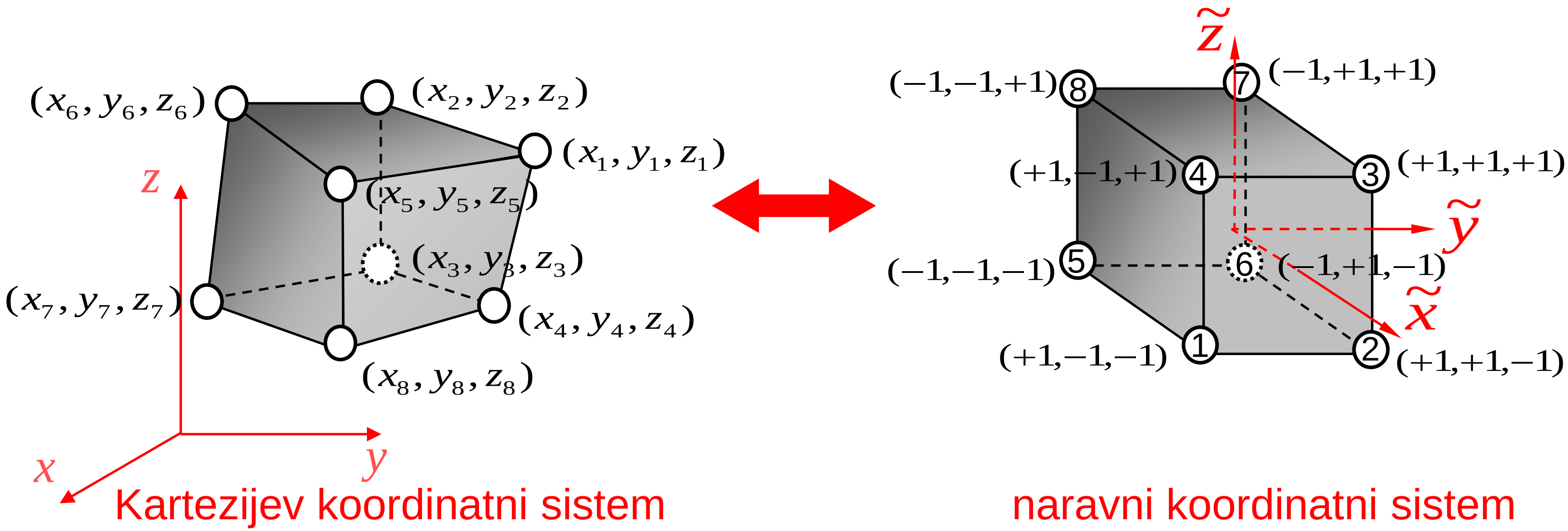
naravni koordinatni sistem

- parcialni odvod temperature po koordinati Kartezijevega koordinatnega sistema

$$\frac{\partial T}{\partial x_{\mathrm{i}}} \approx \frac{\partial \hat{T}}{\partial x_{\mathrm{i}}} = \sum_{\mathrm{j}=1}^{\mathrm{N_v}} T_{\mathrm{j}} \frac{\partial \psi_{\mathrm{j}}(x,y,z)}{\partial x_{\mathrm{i}}} \quad , \quad x_{\mathrm{i}} = x, y, z$$

- povezava med ψ_{j} in $\tilde{\psi}_{\mathrm{j}}$

$$\psi_{\mathrm{j}}(x,y,z) = \psi_{\mathrm{j}}(x(\tilde{x},\tilde{y},\tilde{z}), y(\tilde{x},\tilde{y},\tilde{z}), z(\tilde{x},\tilde{y},\tilde{z})) = \tilde{\psi}_{\mathrm{j}}(\tilde{x},\tilde{y},\tilde{z})$$



- <u>izračun parcialnih odvodov interpolacijskih funkcij ψ_{j} po koordinatah Kartezijevega k. s.</u>

- parcialne odvode interpolacijske funkcije $\tilde{\psi}_{\mathrm{j}}$ po koordinatah naravnega k.s. lahko zapišemo

$$\frac{\partial\tilde{\psi}_{\mathrm{j}}(\tilde{x},\tilde{y},\tilde{z})}{\partial\tilde{x}_{\mathrm{i}}}=\frac{\partial\psi_{\mathrm{j}}(x,y,z)}{\partial x}\frac{\partial x}{\partial\tilde{x}_{\mathrm{i}}}+\frac{\partial\psi_{\mathrm{j}}(x,y,z)}{\partial y}\frac{\partial y}{\partial\tilde{x}_{\mathrm{i}}}+\frac{\partial\psi_{\mathrm{j}}(x,y,z)}{\partial z}\frac{\partial z}{\partial\tilde{x}_{\mathrm{i}}}$$

$$\tilde{x}_{\mathrm{i}}=\tilde{x},\tilde{y},\tilde{z}$$

oziroma v matrični obliki

$$\begin{Bmatrix}\dfrac{\partial\tilde{\psi}_{\mathrm{j}}}{\partial\tilde{x}}\\ \dfrac{\partial\tilde{\psi}_{\mathrm{j}}}{\partial\tilde{y}}\\ \dfrac{\partial\tilde{\psi}_{\mathrm{j}}}{\partial\tilde{z}}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial x}{\partial\tilde{x}} & \dfrac{\partial y}{\partial\tilde{x}} & \dfrac{\partial z}{\partial\tilde{x}}\\ \dfrac{\partial x}{\partial\tilde{y}} & \dfrac{\partial y}{\partial\tilde{y}} & \dfrac{\partial z}{\partial\tilde{y}}\\ \dfrac{\partial x}{\partial\tilde{z}} & \dfrac{\partial y}{\partial\tilde{z}} & \dfrac{\partial z}{\partial\tilde{z}}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial\psi_{\mathrm{j}}}{\partial x}\\ \dfrac{\partial\psi_{\mathrm{j}}}{\partial y}\\ \dfrac{\partial\psi_{\mathrm{j}}}{\partial z}\end{Bmatrix}$$

- parcialne odvode interpolacijske funkcije ψ_j po koordinatah Kartezijevega k.s. lahko sedaj izračunamo na sledeči način

$$\begin{Bmatrix} \dfrac{\partial \psi_j}{\partial x} \\ \dfrac{\partial \psi_j}{\partial y} \\ \dfrac{\partial \psi_j}{\partial z} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \tilde{x}} & \dfrac{\partial y}{\partial \tilde{x}} & \dfrac{\partial z}{\partial \tilde{x}} \\ \dfrac{\partial x}{\partial \tilde{y}} & \dfrac{\partial y}{\partial \tilde{y}} & \dfrac{\partial z}{\partial \tilde{y}} \\ \dfrac{\partial x}{\partial \tilde{z}} & \dfrac{\partial y}{\partial \tilde{z}} & \dfrac{\partial z}{\partial \tilde{z}} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial \tilde{\psi}_j}{\partial \tilde{x}} \\ \dfrac{\partial \tilde{\psi}_j}{\partial \tilde{y}} \\ \dfrac{\partial \tilde{\psi}_j}{\partial \tilde{z}} \end{Bmatrix}$$

- matriko parcialnih odvodov Kartezijevih koordinat po naravnih koordinatah imenujemo Jacobijeva matrika

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \tilde{x}} & \dfrac{\partial y}{\partial \tilde{x}} & \dfrac{\partial z}{\partial \tilde{x}} \\ \dfrac{\partial x}{\partial \tilde{y}} & \dfrac{\partial y}{\partial \tilde{y}} & \dfrac{\partial z}{\partial \tilde{y}} \\ \dfrac{\partial x}{\partial \tilde{z}} & \dfrac{\partial y}{\partial \tilde{z}} & \dfrac{\partial z}{\partial \tilde{z}} \end{bmatrix}$$

$$[J]^{-1} = \begin{bmatrix} \dfrac{\partial x}{\partial \tilde{x}} & \dfrac{\partial y}{\partial \tilde{x}} & \dfrac{\partial z}{\partial \tilde{x}} \\ \dfrac{\partial x}{\partial \tilde{y}} & \dfrac{\partial y}{\partial \tilde{y}} & \dfrac{\partial z}{\partial \tilde{y}} \\ \dfrac{\partial x}{\partial \tilde{z}} & \dfrac{\partial y}{\partial \tilde{z}} & \dfrac{\partial z}{\partial \tilde{z}} \end{bmatrix}^{-1} = \begin{bmatrix} I_{x\tilde{x}} & I_{x\tilde{y}} & I_{x\tilde{z}} \\ I_{y\tilde{x}} & I_{y\tilde{y}} & I_{y\tilde{z}} \\ I_{z\tilde{x}} & I_{z\tilde{y}} & I_{z\tilde{z}} \end{bmatrix}$$

- zahteve, ki jih mora izpolnjevati interpolacijska funkcija ψ

1) interpolacijska funkcija je polinomska funkcija, ki ima vsaj toliko monomov, kot ima KE vozlišč

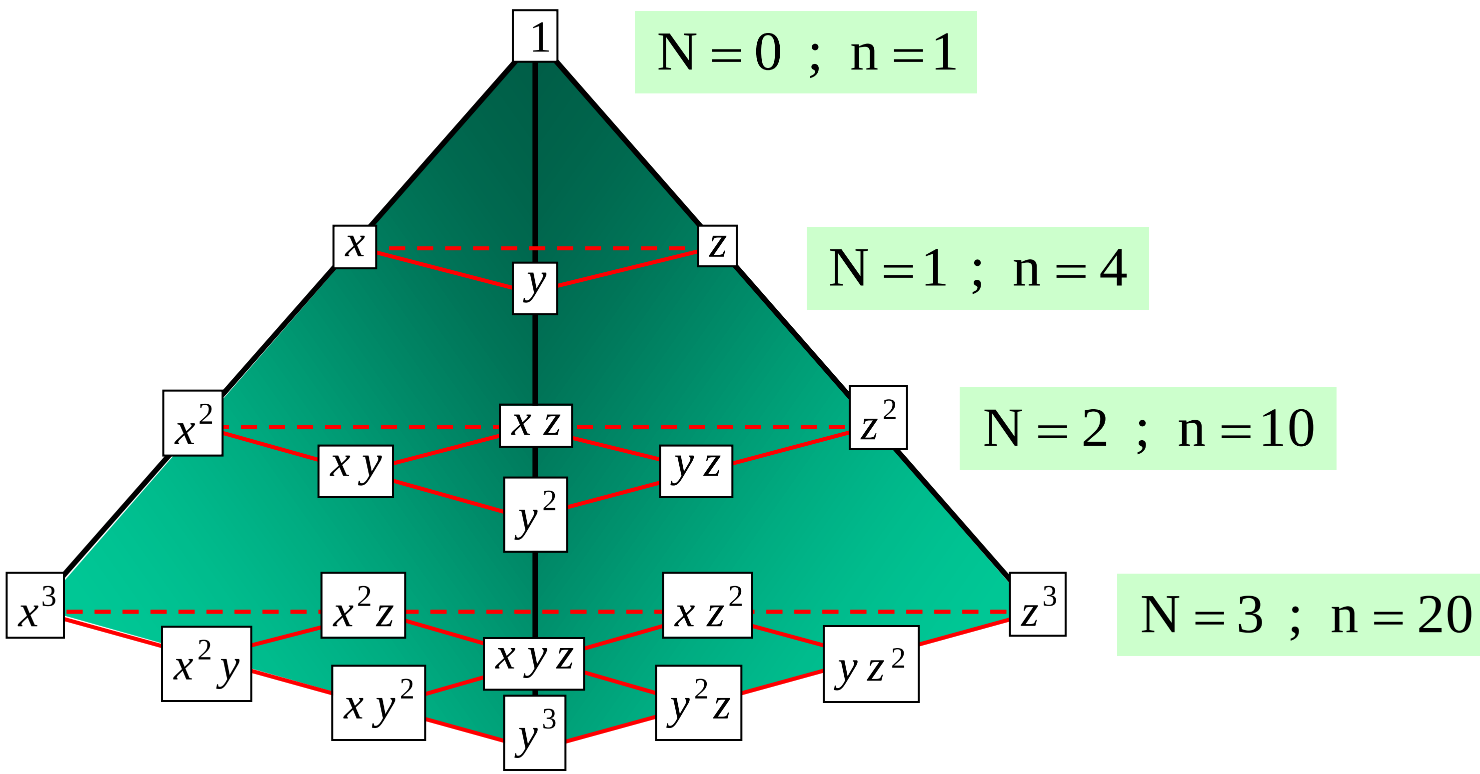
2) monomi, ki nastopajo v interpolacijski funkciji, morajo biti med seboj linearno neodvisni

3) polinomska funkcija mora zagotavljati zvezni prehod polja primarne spremenljivke preko ograje KE, v določenih primerih pa tudi zveznost njenih odvodov

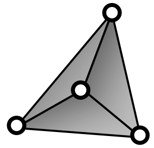
4) polinomska funkcija naj bi izkazovala kompletnost, oziroma v primeru, ko pogoju kompletnosti ni mogoče zadostiti, vsaj geometrijsko izotropnost

- <u>določitev interpolacijske funkcije</u> ψ

1) v trirazsežnem prostoru lahko grafično prikažemo monome, ki nastopajo v kompletni polinomski funkcij $\psi(x,y,z) = P_N(x,y,z)$, v obliki t.i. Pascalovega tetraedra

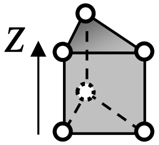


2) oblika interpolacijske funkcije glede na število vozlišč KE



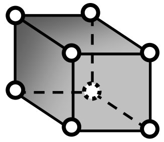
$$\mathrm{N_v} = 4:$$

$$\psi(x,y,z) = P_1(x,y,z) = C_1 1 + C_2\,x + C_3\,y + C_4\,z$$



$$\mathrm{N_v} = 6:$$

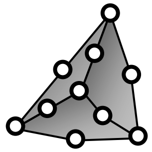
$$\psi(x,y,z) = P_2(x,y,z) = C_1 1 + C_2\,x + C_3\,y + C_4\,z + C_5\,xz + C_6\,yz$$



$$\mathrm{N_v} = 8:$$

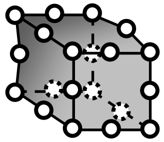
$$\psi(x,y,z) = P_3(x,y,z) = C_1 1 + C_2\,x + C_3\,y + C_4\,z + \\ + C_5\,xy + C_6\,xz + C_7\,yz + C_8\,xyz$$

2) oblika interpolacijske funkcije glede na število vozlišč KE



$$\mathrm{N_v} = 10:$$

$$\psi(x,y,z) = P_2(x,y,z) = C_1 1 + C_2 x + C_3 y + C_4 z + \\ + C_5 x^2 + C_6 y^2 + C_7 z^2 + C_8 xy + C_9 xz + C_{10} yz$$



$$\mathrm{N_v} = 20:$$

$$\psi(x,y,z) = P_3(x,y,z) = C_1 1 + C_2 x + C_3 y + C_4 z + \\ + C_5 x^2 + C_6 y^2 + C_7 z^2 + C_8 xy + C_9 xz + C_{10} yz + \\ + C_{11} x^3 + C_{12} y^3 + C_{13} z^3 + C_{14} x^2 y + C_{15} xy^2 + \\ + C_{16} x^2 z + C_{17} xz^2 + C_{18} y^2 z + C_{19} yz^2 + C_{20} xyz$$

3) za posamezni KE moramo določiti toliko interpolacijskih funkcij ψ_j , kot ima KE vozlišč ($j=1,..,N_v$)

4) posamezna interpolacijska funkcija ψ_j mora imeti v vozliščih KE, s koordinatami (x_i,y_i,z_i), sledeče vrednosti

$$\psi_j(x_i, y_i, z_i) = \begin{cases} 1 \ , & j = i \\ 0 \ , & j \neq i \end{cases} \qquad i, j = 1,.., N_v$$

5) zaradi zagotavljanja možnosti popisa konstantne vrednosti primarne spremenljivke po območju KE, mora biti izpolnjen sledeči pogoj

$$\sum_{j=1}^{N_v} \psi_j(x, y, z) = 1$$