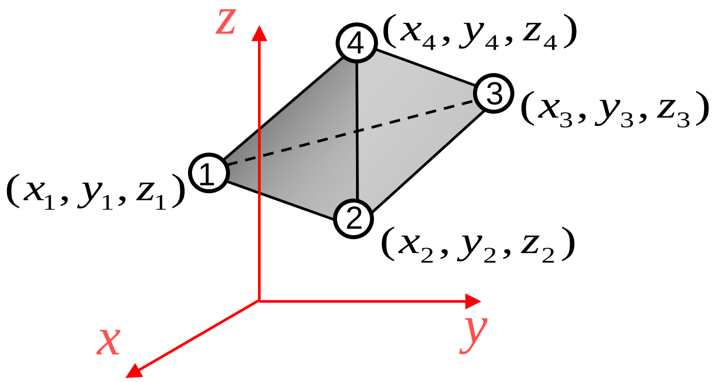
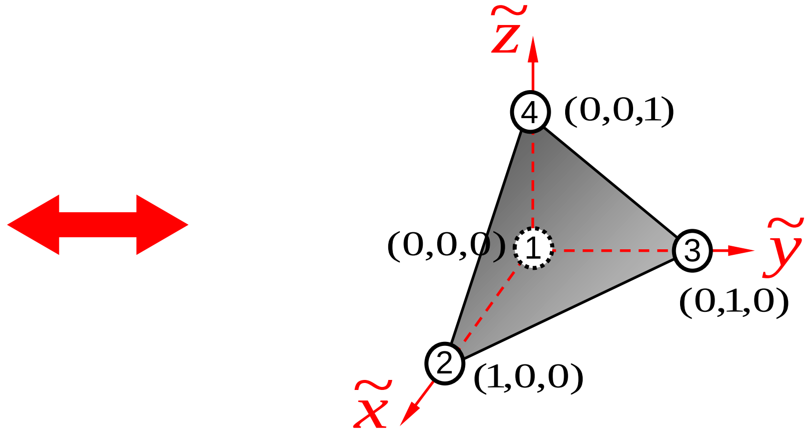
• primer izpeljave interpolacijske funkcije za štiri vozliščni tetraedrični KE:

$$\tilde{\psi}_j(\tilde{x},\tilde{y},\tilde{z}) = C_{1j}\,1 + C_{2j}\,\tilde{x} + C_{3j}\,\tilde{y} + C_{4j}\,\tilde{z}$$
$$j = 1,2,3,4$$



Kartezijev koordinatni sistem

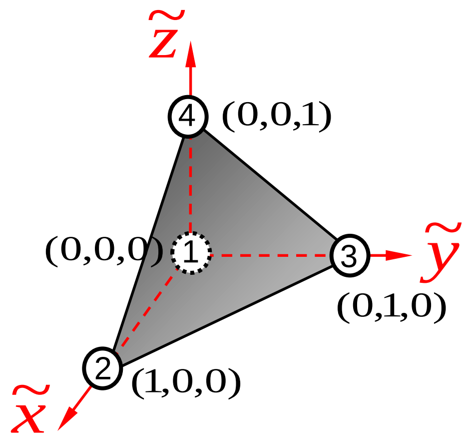


naravni koordinatni sistem

- izračun konstant C_{ik}

$$\begin{bmatrix} 1 & \tilde{x}_1 & \tilde{y}_1 & \tilde{z}_1 \\ 1 & \tilde{x}_2 & \tilde{y}_2 & \tilde{z}_2 \\ 1 & \tilde{x}_3 & \tilde{y}_3 & \tilde{z}_3 \\ 1 & \tilde{x}_4 & \tilde{y}_4 & \tilde{z}_4 \end{bmatrix} \begin{Bmatrix} C_{1j} \\ C_{2j} \\ C_{3j} \\ C_{4j} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} C_{1j} \\ C_{2j} \\ C_{3j} \\ C_{4j} \end{Bmatrix} = \begin{Bmatrix} \delta_{1j} \\ \delta_{2j} \\ \delta_{3j} \\ \delta_{4j} \end{Bmatrix}$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$



naravni koordinatni sistem

- izračun konstant $C_{\mathrm{i}1}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{41} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{41} \end{Bmatrix} = \begin{Bmatrix} +1 \\ -1 \\ -1 \\ -1 \end{Bmatrix}$$

$$\tilde{\psi}_1(\tilde{x},\tilde{y},\tilde{z}) = 1 - \tilde{x} - \tilde{y} - \tilde{z}$$

- izračun konstant $C_{\mathrm{i}2}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} C_{12} \\ C_{22} \\ C_{32} \\ C_{42} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} C_{12} \\ C_{22} \\ C_{32} \\ C_{42} \end{Bmatrix} = \begin{Bmatrix} 0 \\ +1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\tilde{\psi}_2(\tilde{x},\tilde{y},\tilde{z}) = \tilde{x}$$

- izračun konstant C_{i3}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{33} \\ C_{43} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{33} \\ C_{43} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ +1 \\ 0 \end{Bmatrix}$$

$$\tilde{\psi}_3(\tilde{x},\tilde{y},\tilde{z}) = \tilde{y}$$

- izračun konstant C_{i4}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} C_{14} \\ C_{24} \\ C_{34} \\ C_{44} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} C_{14} \\ C_{24} \\ C_{34} \\ C_{44} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ +1 \end{Bmatrix}$$

$$\tilde{\psi}_4(\tilde{x},\tilde{y},\tilde{z}) = \tilde{z}$$

- <u>matrični zapis enačbe KE za problem ustaljenega prevoda toplote v volumnu</u>

- za posamezni KE moramo zapisati toliko enačb, kolikor ima KE vozlišč (v vozlišču KE je neznana <u>ena primarna spremenljivka</u> – <u>temperatura</u>)

$$k \int_{\Omega} \left[\frac{\partial T}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial T}{\partial z}\frac{\partial f}{\partial z} \right] \mathrm{d}\Omega =$$

$$= -\int_{\Gamma} [\, q_{\mathrm{x}}\, n_{\mathrm{x}} + q_{\mathrm{y}}\, n_{\mathrm{y}} + q_{\mathrm{z}}\, n_{\mathrm{z}}]\; f\, \mathrm{d}\Gamma + \int_{\Omega} Q\, f\, \mathrm{d}\Omega$$

$$f = \{\psi_{\mathrm{I}}(x, y, z)\} \;, \quad \mathrm{I} = 1,.., \mathrm{N_v}$$

- sistem $\mathrm{N_v}$ enačb za posamezni KE

$$k\int_{\Omega}\left[\frac{\partial T}{\partial x}\frac{\partial \psi_{\mathrm{I}}}{\partial x}+\frac{\partial T}{\partial y}\frac{\partial \psi_{\mathrm{I}}}{\partial y}+\frac{\partial T}{\partial z}\frac{\partial \psi_{\mathrm{I}}}{\partial z}\right]\mathrm{d}\Omega =$$

$$=-\int_{\Gamma}[q_{\mathrm{x}}\,n_{\mathrm{x}}+q_{\mathrm{y}}\,n_{\mathrm{y}}+q_{\mathrm{z}}\,n_{\mathrm{z}}]\,\psi_{\mathrm{I}}\,\mathrm{d}\Gamma+\int_{\Omega}Q\,\psi_{\mathrm{I}}\,\mathrm{d}\Omega \quad , \quad \mathrm{I}=1,..,\mathrm{N_v}$$

- upoštevajoč zvezo

$$\frac{\partial T}{\partial x_{\mathrm{i}}}\approx\frac{\partial \hat{T}}{\partial x_{\mathrm{i}}}=\sum_{\mathrm{j}=1}^{\mathrm{N_v}}T_{\mathrm{j}}\frac{\partial \psi_{\mathrm{j}}(x,y,z)}{\partial x_{\mathrm{i}}} \quad , \quad x_{\mathrm{i}}=x,y,z$$

lahko sistem $\mathrm{N_v}$ enačb zapišemo

$$k\int_{\Omega}\left[\left\{\sum_{\mathrm{j}=1}^{\mathrm{N_v}}T_{\mathrm{j}}\frac{\partial \psi_{\mathrm{j}}}{\partial x}\right\}\frac{\partial \psi_{\mathrm{I}}}{\partial x}+\left\{\sum_{\mathrm{j}=1}^{\mathrm{N_v}}T_{\mathrm{j}}\frac{\partial \psi_{\mathrm{j}}}{\partial y}\right\}\frac{\partial \psi_{\mathrm{I}}}{\partial y}+\left\{\sum_{\mathrm{j}=1}^{\mathrm{N_v}}T_{\mathrm{j}}\frac{\partial \psi_{\mathrm{j}}}{\partial z}\right\}\frac{\partial \psi_{\mathrm{I}}}{\partial z}\right]\mathrm{d}\Omega =$$

$$=-\int_{\Gamma}[q_{\mathrm{x}}\,n_{\mathrm{x}}+q_{\mathrm{y}}\,n_{\mathrm{y}}+q_{\mathrm{z}}\,n_{\mathrm{z}}]\,\psi_{\mathrm{I}}\,\mathrm{d}\Gamma+\int_{\Omega}Q\,\psi_{\mathrm{I}}\,\mathrm{d}\Omega \quad , \quad \mathrm{I}=1,..,\mathrm{N_v}$$

- matrični zapis leve strani sistema enačb

$$k\int_{\Omega}\left[\left\{\sum_{j=1}^{N_v} T_j \frac{\partial\psi_j}{\partial x}\right\}\frac{\partial\psi_I}{\partial x}+\left\{\sum_{j=1}^{N_v} T_j \frac{\partial\psi_j}{\partial y}\right\}\frac{\partial\psi_I}{\partial y}+\left\{\sum_{j=1}^{N_v} T_j \frac{\partial\psi_j}{\partial z}\right\}\frac{\partial\psi_I}{\partial z}\right]d\Omega =$$

$$= k\int_{\Omega}\sum_{j=1}^{N_v}\left[\left\{\frac{\partial\psi_j}{\partial x}\frac{\partial\psi_I}{\partial x}+\frac{\partial\psi_j}{\partial y}\frac{\partial\psi_I}{\partial y}+\frac{\partial\psi_j}{\partial z}\frac{\partial\psi_I}{\partial z}\right\}T_j\right]d\Omega =$$

$$= k\sum_{j=1}^{N_v}\left[\int_{\Omega}\left\{\frac{\partial\psi_I}{\partial x}\frac{\partial\psi_j}{\partial x}+\frac{\partial\psi_I}{\partial y}\frac{\partial\psi_j}{\partial y}+\frac{\partial\psi_I}{\partial z}\frac{\partial\psi_j}{\partial z}\right\}d\Omega\right]T_j =$$

$$= k\,[M_k]\{T\} =$$

$$= k\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1N_v} \\ M_{21} & M_{22} & \cdots & M_{2N_v} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N_v 1} & M_{N_v 2} & \cdots & M_{N_v N_v} \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{N_v} \end{Bmatrix}$$

- matrični zapis desne strani sistema enačb

$$-\int_{\Gamma} [\, q_{\mathrm{x}}\, n_{\mathrm{x}} + q_{\mathrm{y}}\, n_{\mathrm{y}} + q_{\mathrm{z}}\, n_{\mathrm{z}} \,]\, \psi_{\mathrm{I}}\, \mathrm{d}\Gamma + \int_{\Omega} Q\, \psi_{\mathrm{I}}\, \mathrm{d}\Omega =$$

$$= \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\mathrm{N_v}} \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{\mathrm{N_v}} \end{Bmatrix}$$

- matrični zapis sistema $\mathrm{N_v}$ enačb za posamezni volumski KE

$$k\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1\mathrm{N_v}} \\ M_{21} & M_{22} & \cdots & M_{2\mathrm{N_v}} \\ \vdots & \vdots & \ddots & \vdots \\ M_{\mathrm{N_v}1} & M_{\mathrm{N_v}2} & \cdots & M_{\mathrm{N_v}\mathrm{N_v}} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{\mathrm{N_v}} \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\mathrm{N_v}} \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{\mathrm{N_v}} \end{Bmatrix}$$

$$M_{\mathrm{IJ}} = \int_\Omega \left\{ \frac{\partial \psi_\mathrm{I}}{\partial x} \frac{\partial \psi_\mathrm{J}}{\partial x} + \frac{\partial \psi_\mathrm{I}}{\partial y} \frac{\partial \psi_\mathrm{J}}{\partial y} + \frac{\partial \psi_\mathrm{I}}{\partial z} \frac{\partial \psi_\mathrm{J}}{\partial z} \right\} \mathrm{d}\Omega = M_{\mathrm{JI}}$$

$$q_\mathrm{I} = -\int_\Gamma [\, q_\mathrm{x}\, n_\mathrm{x} + q_\mathrm{y}\, n_\mathrm{y} + q_\mathrm{z}\, n_\mathrm{z} \,]\, \psi_\mathrm{I}\, \mathrm{d}\Gamma$$

$$Q_\mathrm{I} = \int_\Omega Q\, \psi_\mathrm{I}\, \mathrm{d}\Omega$$

- prehod iz <u>Kartezijevega koordinatnega sistema</u> v <u>naravni koordinatni sistem</u>

$$M_{\mathrm{IJ}} = \int_{\Omega} \left\{ \frac{\partial \psi_{\mathrm{I}}}{\partial x} \frac{\partial \psi_{\mathrm{J}}}{\partial x} + \frac{\partial \psi_{\mathrm{I}}}{\partial y} \frac{\partial \psi_{\mathrm{J}}}{\partial y} + \frac{\partial \psi_{\mathrm{I}}}{\partial z} \frac{\partial \psi_{\mathrm{J}}}{\partial z} \right\} \mathrm{d}\Omega$$

$$\frac{\partial \psi_{\mathrm{I}}}{\partial x} = I_{x\tilde{x}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{x}} + I_{x\tilde{y}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{y}} + I_{x\tilde{z}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{z}} = \tilde{F}_{\mathrm{xI}}(\tilde{x}, \tilde{y}, \tilde{z})$$

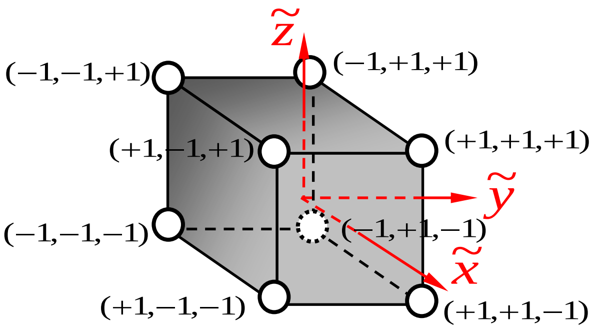
$$\frac{\partial \psi_{\mathrm{I}}}{\partial y} = I_{y\tilde{x}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{x}} + I_{y\tilde{y}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{y}} + I_{y\tilde{z}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{z}} = \tilde{F}_{\mathrm{yI}}(\tilde{x}, \tilde{y}, \tilde{z})$$

$$\frac{\partial \psi_{\mathrm{I}}}{\partial z} = I_{z\tilde{x}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{x}} + I_{z\tilde{y}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{y}} + I_{z\tilde{z}} \frac{\partial \tilde{\psi}_{\mathrm{I}}}{\partial \tilde{z}} = \tilde{F}_{\mathrm{zI}}(\tilde{x}, \tilde{y}, \tilde{z})$$

$$M_{\mathrm{IJ}} = \int_{\tilde{\Omega}} \left\{ \tilde{F}_{\mathrm{xI}}\, \tilde{F}_{\mathrm{xJ}} + \tilde{F}_{\mathrm{yI}}\, \tilde{F}_{\mathrm{yJ}} + \tilde{F}_{\mathrm{zI}}\, \tilde{F}_{\mathrm{zJ}} \right\} |J|\, \mathrm{d}\tilde{\Omega} =$$

$$= \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} \left\{ \tilde{F}_{\mathrm{xI}}\, \tilde{F}_{\mathrm{xJ}} + \tilde{F}_{\mathrm{yI}}\, \tilde{F}_{\mathrm{yJ}} + \tilde{F}_{\mathrm{zI}}\, \tilde{F}_{\mathrm{zJ}} \right\} |J|\, \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{y}\, \mathrm{d}\tilde{z}$$

$$\mathrm{d}\Omega = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = |J|\, \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{y}\, \mathrm{d}\tilde{z} = |J|\, \mathrm{d}\tilde{\Omega}$$



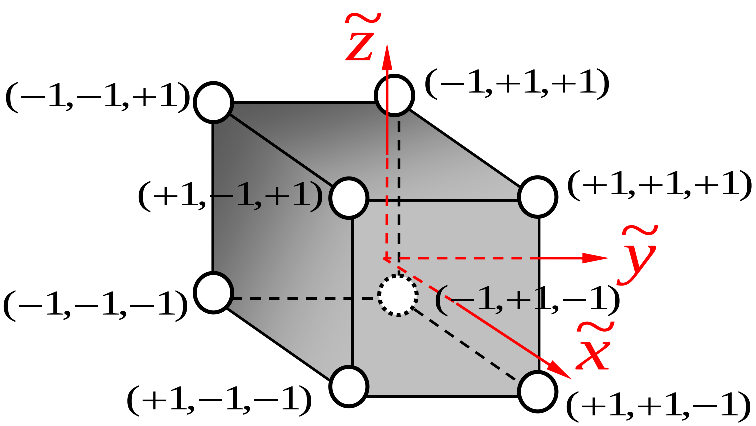
- prehod iz Kartezijevega koordinatnega sistema v naravni koordinatni sistem

$$q_I = -\int_{\Gamma} [q_x n_x + q_y n_y + q_z n_z]\, \psi_I \, d\Gamma = -\int_{\tilde{\Gamma}} [q_x n_x + q_y n_y + q_z n_z]\, \tilde{\psi}_I \, |j| \, d\tilde{\Gamma} =$$

$$= \left\{ -\int_{-1}^{+1}\int_{-1}^{+1} [q_x n_x + q_y n_y + q_z n_z]\, \tilde{\psi}_I \, |j| \, d\tilde{x}\, d\tilde{y} \right\}_{\tilde{z}=+1} + \left\{ -\int_{-1}^{+1}\int_{-1}^{+1} [q_x n_x + q_y n_y + q_z n_z]\, \tilde{\psi}_I \, |j| \, d\tilde{x}\, d\tilde{y} \right\}_{\tilde{z}=-1} +$$

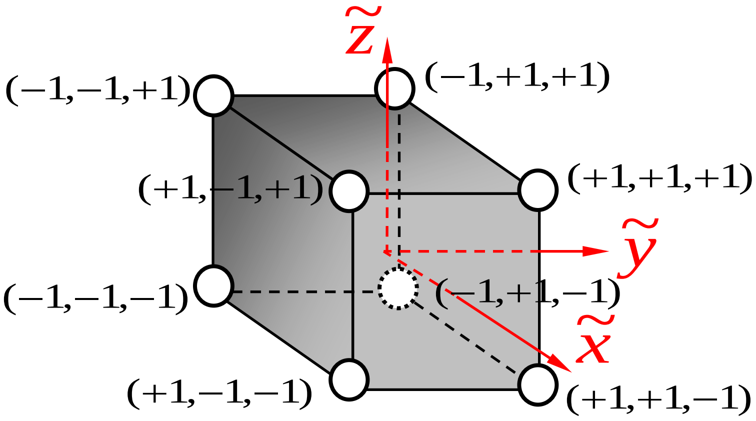
$$+ \left\{ -\int_{-1}^{+1}\int_{-1}^{+1} [q_x n_x + q_y n_y + q_z n_z]\, \tilde{\psi}_I \, |j| \, d\tilde{x}\, d\tilde{z} \right\}_{\tilde{y}=+1} + \left\{ -\int_{-1}^{+1}\int_{-1}^{+1} [q_x n_x + q_y n_y + q_z n_z]\, \tilde{\psi}_I \, |j| \, d\tilde{x}\, d\tilde{z} \right\}_{\tilde{y}=-1} +$$

$$+ \left\{ -\int_{-1}^{+1}\int_{-1}^{+1} [q_x n_x + q_y n_y + q_z n_z]\, \tilde{\psi}_I \, |j| \, d\tilde{y}\, d\tilde{z} \right\}_{\tilde{x}=+1} + \left\{ -\int_{-1}^{+1}\int_{-1}^{+1} [q_x n_x + q_y n_y + q_z n_z]\, \tilde{\psi}_I \, |j| \, d\tilde{y}\, d\tilde{z} \right\}_{\tilde{x}=-1}$$



- prehod iz Kartezijevega koordinatnega sistema v naravni koordinatni sistem

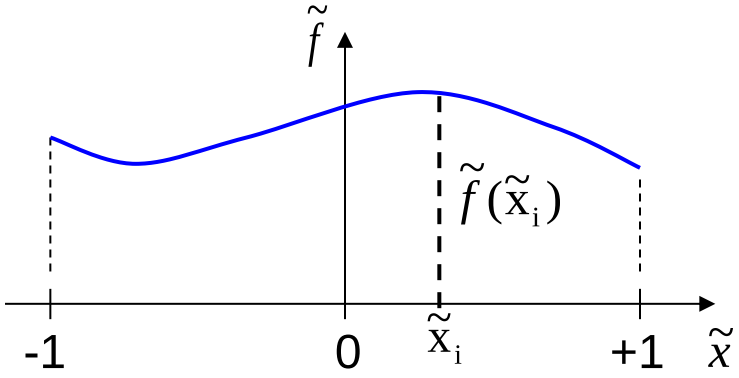
$$Q_{\mathrm{I}} = \int_{\Omega} Q\,\psi_{\mathrm{I}}\,\mathrm{d}\Omega = \int_{\tilde{\Omega}} Q\,\tilde{\psi}_{\mathrm{I}}\,|J|\,\mathrm{d}\tilde{\Omega} = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} Q\,\tilde{\psi}_{\mathrm{I}}\,|J|\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{z}$$



- numerično integriranje: Gaussova integracijska formula

- Gaussova integracijska formula

$$I = \int_{x_{sp}}^{x_{zg}} f(x)\,\mathrm{d}x = \int_{-1}^{+1} \tilde{f}(\tilde{x})\,\mathrm{d}\tilde{x} \approx \sum_{i=1}^{m} w_i\, \tilde{f}(\tilde{x}_i)$$



$$I = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \cdots + \frac{2}{2k+1}a_k \approx$$

$$\approx \sum_{i=1}^{m} w_i \left\{ a_0 + a_1 \tilde{x}_i + a_2 \tilde{x}_i^2 + a_3 \tilde{x}_i^3 + \cdots + a_n \tilde{x}_i^n \right\}, \; k \leq n/2$$

$$\sum_{i=1}^{m} w_i a_0 = 2a_0 \;\Rightarrow\; \sum_{i=1}^{m} w_i = 2$$

$$\sum_{i=1}^{m} w_i \tilde{x}_i a_1 = 0a_1 \;\Rightarrow\; \sum_{i=1}^{m} w_i \tilde{x}_i = 0$$

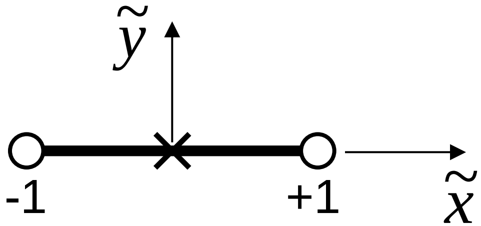
$$\sum_{i=1}^{m} w_i \tilde{x}_i^2 a_2 = \frac{2}{3}a_2 \;\Rightarrow\; \sum_{i=1}^{m} w_i \tilde{x}_i^2 = \frac{2}{3}$$

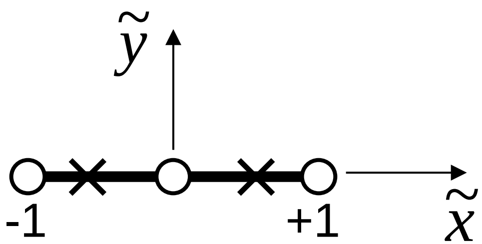
$$\sum_{i=1}^{m} w_i \tilde{x}_i^3 a_3 = 0a_3 \;\Rightarrow\; \sum_{i=1}^{m} w_i \tilde{x}_i^3 = 0$$

$$\sum_{i=1}^{m} w_i \tilde{x}_i^4 a_4 = \frac{2}{5}a_4 \;\Rightarrow\; \sum_{i=1}^{m} w_i \tilde{x}_i^4 = \frac{2}{5}$$

$$I = \int_{x_{sp}}^{x_{zg}} f(x)\,\mathrm{d}x = \int_{-1}^{+1} \tilde{f}(\tilde{x})\,\mathrm{d}\tilde{x} \approx \sum_{i=1}^{m} w_i \, \tilde{f}(\tilde{x}_i)$$

- lega integracijskih točk v primeru 1D KE:



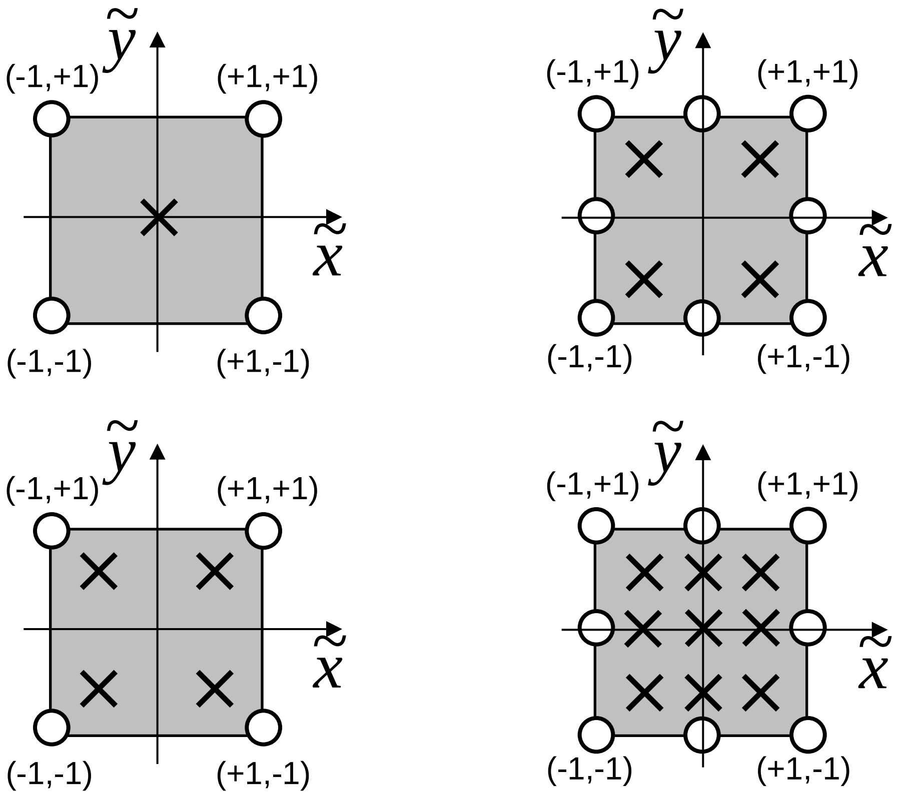


m	i	w_i	$\tilde{x}_i$
1	1	2,000000	0,000000
2	1	1,000000	+0,577350
	2	1,000000	-0,577350
3	1	0,888889	0,000000
	2	0,555556	+0,774597
	3	0,555556	-0,774597
4	1	0,652145	+0,339981
	2	0,652145	-0,339981
	3	0,347855	+0,861136
	4	0,347855	-0,861136

- numerični izračun integrala po dveh spremenljivkah z Gaussovo integracijsko formulo

$$I = \int_{-1}^{+1}\int_{-1}^{+1} \tilde{f}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y} \approx \sum_{j=1}^{m}\sum_{i=1}^{m} w_j\, w_i\, \tilde{f}(\tilde{x}_i,\tilde{y}_j)$$

- lega integracijskih točk v primeru 2D KE:



- numerični izračun integrala po treh spremenljivkah z Gaussovo integracijsko formulo

$$I = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \tilde{f}(\tilde{x},\tilde{y},\tilde{z})\, d\tilde{x}\, d\tilde{y}\, d\tilde{z} \approx \sum_{k=1}^{m}\sum_{j=1}^{m}\sum_{i=1}^{m} w_k\, w_j\, w_i\, \tilde{f}(\tilde{x}_i,\tilde{y}_j,\tilde{z}_k)$$

- lega integracijskih točk v primeru 3D KE: